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On a Surface-Related Multi-Layer Shell Theory and its Application to Compound Structures

The paper is concerned with the analytical formulation of a multi-layer surface-related shell theory and with the basic concepts of the used contact formulation.

1. Single-Layer Shell Theory

For the description of the differential geometry of the undeformed shell continuum a parametrization by curvilinear, convected coordinates Θ^α and the normal coordinate Θ^3 is chosen as usual. With this parametrization every point of the shell continuum is characterized by a position vector $\mathbf{X} = \overline{\mathbf{X}} + \Theta^3 \mathbf{A}_3$, wherefrom the covariant base vectors $\mathbf{G}_\alpha = \mathbf{A}_\alpha + \Theta^3 \mathbf{A}_{3,\alpha} = \left(\delta_\alpha^\lambda - \Theta^3 B_\alpha^\lambda\right) \mathbf{A}_\lambda$ and $\mathbf{G}_3 = \mathbf{A}_3$ with δ_α^λ as Kronecker symbol and B_α^λ as components of the curvature tensor can be calculated by partial differentiation. Considering the surface-relation a domain $0 \le \Theta^3 \le H(\Theta^\alpha)$ for the coordinate Θ^3 is defined with H as shell thickness.
The displacement field $\overline{\mathbf{u}}$ of the shell continuum is approximated by the infinite series

$$\overline{\mathbf{u}}\left(\Theta^1, \Theta^2, \Theta^3\right) = \sum_{l=0}^{\infty} {}_l\Omega \, {}_l\overline{\mathbf{u}}\left(\Theta^1, \Theta^2\right) , \tag{1}$$

where ${}_l\overline{\mathbf{u}}$ is the displacement field of the l-th director of the shell and ${}_0\overline{\mathbf{u}}$ the displacement field of the reference surface, cp. [1]. A chosen complete base function system ${}_l\Omega = \left(\Theta^3\right)^l$ represents an ansatz in the thickness-direction. If we introduce the displacement field (1) into the Green-Lagrange strain tensor $\mathbf{E} = E_{ij}\, \mathbf{G}^i \otimes \mathbf{G}^j = \frac{1}{2}(\overline{\mathbf{u}}_{,i} \cdot \mathbf{G}_j + \overline{\mathbf{u}}_{,j} \cdot \mathbf{G}_i + \overline{\mathbf{u}}_{,i} \cdot \overline{\mathbf{u}}_{,j})\mathbf{G}^i \otimes \mathbf{G}^j$, we get the components

$$E_{\alpha\beta} = \frac{1}{2} \sum_{l=0}^{\infty} \left({}_l a_{\alpha\beta}\, {}_l\Omega - \Theta^3\, {}_l b_{\alpha\beta}\, {}_l\Omega + \sum_{k=0}^{\infty} \left({}_{kl} c_{\alpha\beta}\, {}_k\Omega\, {}_l\Omega \right) \right) \tag{2}$$

$$E_{3\alpha} = \frac{1}{2} \sum_{l=0}^{\infty} \left({}_l d_{\alpha 3}\, {}_l\Omega + {}_l e_{\alpha 3}\, {}_l\Omega_{,3} - \Theta^3\, {}_l f_{\alpha 3}\, {}_l\Omega_{,3} + \sum_{k=0}^{\infty} \left({}_{kl} g_{\alpha 3}\, {}_k\Omega\, {}_l\Omega_{,3} \right) \right) \tag{3}$$

$$E_{33} = \frac{1}{2} \sum_{l=0}^{\infty} \left({}_l h_{33}\, {}_l\Omega_{,3} + \sum_{k=0}^{\infty} \left({}_{kl} i_{33}\, {}_k\Omega_{,3}\, {}_l\Omega_{,3} \right) \right) \tag{4}$$

of the strain tensor with the substrain measures

$$\begin{aligned}
{}_l a_{\alpha\beta} &= {}_l\mathbf{u}_{,\alpha} \cdot \mathbf{A}_\beta + {}_l\mathbf{u}_{,\beta} \cdot \mathbf{A}_\alpha & {}_l d_{\alpha 3} &= {}_l\mathbf{u}_{,\alpha} \cdot \mathbf{A}_3 & {}_{kl} g_{\alpha 3} &= {}_k\mathbf{u}_{,\alpha} \cdot {}_l\mathbf{u} \\
{}_l b_{\alpha\beta} &= \mathbf{A}_\lambda \cdot \left(B_\beta^\lambda\, {}_l\mathbf{u}_{,\alpha} + B_\alpha^\lambda\, {}_l\mathbf{u}_{,\beta} \right) & {}_l e_{\alpha 3} &= {}_l\mathbf{u} \cdot \mathbf{A}_\alpha & {}_l h_{33} &= 2\, {}_l\mathbf{u} \cdot \mathbf{A}_3 \\
{}_{kl} c_{\alpha\beta} &= {}_k\mathbf{u}_{,\alpha} \cdot {}_l\mathbf{u}_{,\beta} & {}_l f_{\alpha 3} &= B_\alpha^\lambda \mathbf{A}_\lambda \cdot {}_l\mathbf{u} & {}_{kl} i_{33} &= {}_k\mathbf{u} \cdot {}_l\mathbf{u} \,.
\end{aligned} \tag{5}$$

Because of the nonlinearity and the chosen ansatz we get double sums in (2) to (4).
On the basis of the chosen kinematics it is possible to use three dimensional material laws which are based on the Green-Lagrange strain tensor. In this theory a linear elastic, orthotropic material is used. It is characterized by $\mathbf{S} = \underline{\mathbf{C}} : \mathbf{E}$, whereby $\mathbf{S}$ is the Second Piola-Kirchhoff stress tensor and $\underline{\mathbf{C}} = C_m^{ijkl} \mathbf{M}_i \otimes \mathbf{M}_j \otimes \mathbf{M}_k \otimes \mathbf{M}_l$ $= C^{ijkl} \mathbf{G}_i \otimes \mathbf{G}_j \otimes \mathbf{G}_k \otimes \mathbf{G}_l$ is the tensor of elasticity. The components C_m^{ijkl} belong to an orthogonal material coordinate system $\mathbf{M}_i$ characterizing the principle directions of the material behavior. To use the material law in the internal virtual work, it is helpful to transform the tensor into the basis system $\mathbf{G}_i$.

2. Multi-Layer Shell Theory

The multi-layer kinematics is derived analogically to the represented single-layer-kinematics. The displacement $\overline{\mathbf{u}}_L$

of each layer L is again approximated by the infinite series as before, where ${}_0\overline{\mathbf{u}}_L$ is the displacement vector of the L-th layer reference surface. If we introduce the condition $\overline{\mathbf{u}}_L\left(\Theta_L^3=0\right)=\overline{\mathbf{u}}_{L-1}\left(\Theta_{L-1}^3=H_{L-1}\right)$ of a C_0-continuity of the displacement field across the layers, we get the displacement field

$$\overline{\mathbf{u}}_L\left(\Theta^\alpha,\Theta_L^3\right)={}_0\overline{\mathbf{u}}_1+\sum_{K=1}^{L-1}\sum_{l=1}^{d}\left(H_K\right)^l\,{}_l\overline{\mathbf{u}}_K+\sum_{l=1}^{d}\left(\Theta_L^3\right)^l\,{}_l\overline{\mathbf{u}}_L \tag{6}$$

for the shell continuum of the L-th layer. Only the displacement ${}_0\overline{\mathbf{u}}_1$ of the reference surface of the first layer remains in the equation. The second part of the equation represents the deformation of the shell continuum between the first and the L-th reference surface and the last part represents the deformation of the L-th layer itself.
If we drop the condition of C_0-continuity, we get the displacement field

$$\overline{\mathbf{u}}_L\left(\Theta^\alpha,\Theta_L^3\right)={}_0\overline{\mathbf{u}}_L\left(\Theta^\alpha\right)+\sum_{l=1}^{d}\left(\Theta_L^3\right)^l\,{}_l\overline{\mathbf{u}}_L\left(\Theta^\alpha\right)\,. \tag{7}$$

Different to the kinematics (6) it depends on the displacement field of the reference surface of the same layer.
To ensure the interconnection of the layers compound conditions are introduced analogously to the contact mechanics. Therefore it is necessary to define the contact stress vector

$$\mathbf{S}_L=-\mathbf{S}_{L-1}=S_L^{i3}\mathbf{A}_i=\mathbf{S}_{\mathrm{t}\,L}+\mathbf{S}_{\mathrm{n}\,L}\,,\quad \mathbf{S}_{\mathrm{t}\,L}=S_L^{\alpha 3}\mathbf{A}_\alpha\,,\quad \mathbf{S}_{\mathrm{n}\,L}=S_{\mathrm{n}\,L}\,\mathbf{A}_3 \tag{8}$$

and the relative displacement field

$$\Delta\overline{\mathbf{u}}_L=\overline{\mathbf{u}}_L\left(\Theta_L^3=0\right)-\overline{\mathbf{u}}_{L-1}\left(\Theta_{L-1}^3=H_{L-1}\right)=\Delta\overline{u}_{i\,L}\mathbf{A}^i=\Delta\overline{\mathbf{u}}_{\mathrm{t}\,L}+\Delta\overline{\mathbf{u}}_{\mathrm{n}\,L} \tag{9}$$

across the L-th and $(L-1)$-th layer. The characteristic quantities for the decision between contact and uplifting (delamination in normal direction) are the compound condition V and the gap function g_{n}:

$$V=S_{\mathrm{n}\,L}-S_{\mathrm{n}\,\max}\,,\qquad g_{\mathrm{n}}=\overline{u}_{3\,L}-\overline{u}_{3\,L-1}=\Delta\overline{u}_{3\,L}, \tag{10}$$

where $V\leq 0$ and $g_{\mathrm{n}}=0$ at a GAUSS point of the L-th layer in the case of contact and ${}_0\overline{\mathbf{u}}_L\left(\Theta^\alpha\right)$ is coupled with the layer above. In the case of uplifting $g_{\mathrm{n}}\geq 0$ and $S_{\mathrm{n}\,L}=0$ and the vector ${}_0\overline{\mathbf{u}}_L\left(\Theta^\alpha\right)$ supplies three degrees of freedom. Thus the impenetrability of the layers is realized by the kinematical coupling of the layers.
The description of the tangential compound is very similar. The characteristic quantity for the decision between sticking and sliding (delamination in tangential direction) are the compound condition R and the slip vector $\mathbf{g}_{\mathrm{t}}$:

$$R=\left|\mathbf{S}_{\mathrm{t}\,L}\right|-S_{\mathrm{t}\,\max}\,,\qquad \mathbf{g}_{\mathrm{t}}=\Delta\overline{\mathbf{u}}_{\mathrm{t}\,L}, \tag{11}$$

where $R\leq 0$ and $|g_{\mathrm{t}}|=0$ for the L-th layer in the case of sticking in location of a GAUSS point. All of the three components of the vector ${}_0\overline{\mathbf{u}}_L\left(\Theta^\alpha\right)$ are coupled with the layer above. In the case of sliding the tangential stress vector

$$\mathbf{S}_{\mathrm{t}\,L}\left(\mathbf{g}_{\mathrm{t}}\right)=-\frac{\mathbf{g}_{\mathrm{t}}}{\left|\mathbf{g}_{\mathrm{t}}\right|}\left(\mu\,S_{\mathrm{n}\,L}+k\left(\mathbf{g}_{\mathrm{t}}\right)\right) \tag{12}$$

is a function of the slip vector $\mathbf{g}_{\mathrm{t}}$ (11). The first part results from COULOMB's friction law and the second part introduces an additional stiffness like adhesion. The tangential components of the vector ${}_0\overline{\mathbf{u}}_L\left(\Theta^\alpha\right)$ supply two degrees of freedom and only its normal component is coupled with the layer below.

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3. References

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